Master Thesis:

*Multi-period Constrained Portfolio Optimization Using Conditional Value at Risk*

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Abstract

This thesis investigates the problem of asset allocation when making long term investments. The aim of this work is to describe the relationship between the short term risk aversion of most investors and the goal of a good risk-gain balance over long term. This study performs portfolio optimizations using Conditional Value at Risk (CVaR). It starts by solving a single period optimization problem and shows results concerning the dependence of the efficient frontier on the investment horizon when using a buy-and-hold strategy. The central point of this work is the generalization of the CVaR portfolio optimization method for the case of a dual objective investor. It is described how to account simultaneously for long and short term views on market behavior by constructing constrained long-term efficient frontiers. The constraints correspond to maximum risk levels imposed using return distributions implied by the short term capital market assumptions. The method devised is best suited for constructing, in a single quantitative optimization step, both strategic and tactical asset allocation of a private investor.

Keywords: Strategic Asset Allocation, Conditional Value at Risk, Multiperiod Portfolio Optimization, Monte Carlo Simulation

JEL Classification: C15, G11
Executive Summary

The main assumption underlying the most common portfolio optimization procedure is that the mean and variance of the expected return distributions are all that is required in order to choose the best asset allocation. This is true if the returns are distributed according to a normal distribution, but it is not true in general. In the last decade, a whole new field of interest has emerged with the introduction of the notion of Value at Risk (VAR). The idea behind the use of VAR is to encapsulate into a single number all the information about the possible portfolio losses implied by the tail of the return distribution, even in the case when this distribution is not normal. However, portfolio optimization based on VAR has a major drawback because it does not always provide investors with an incentive to diversify their investment across different asset classes. As an alternative to VAR, a better risk measure has recently emerged: the Conditional Value at Risk (CVaR). Introduced only a few years ago, CVaR has gained increasing acceptance in the academic community as the method of choice for portfolio optimizations that need to go beyond the simple mean-variance framework. This method is highly flexible and allows for an easy implementation using any standard linear programming package. Very recently the CVaR optimization method has also started to trickle down from academic studies towards practical use within the financial industry.

In this work, I explore a new facet of the long-term investment problem. My aim is to describe the relationship between the short term risk aversion of most investors and the goal of a good risk-gain balance over the long term. I start by solving a single period optimization problem and show results concerning the dependence of the efficient frontier on the investment horizon when using a buy-and-hold strategy. The central point of this work is the generalization of CVaR portfolio optimization for the case of a dual objective investor. I demonstrate how one can simultaneously account for the long- and short-term views regarding the market behavior. I describe the method of building constrained long-term efficient frontiers, where the constraints correspond to short term market assumptions. The method is very general and could be extended to a truly multiperiod problem. However, the present two-period form (short and long term) allows one to obtain, in a single quantitative optimization step, both tactical and strategic asset allocations. The area where this procedure could be best applied is the design of custom investment solutions within the framework of the private banking industry.
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1. Introduction

Portfolio choice theory has been the quiet backwater of modern finance for a long period of time. Although the mean-variance portfolio analysis of Markowitz (1956) is one of the seminal works of modern finance theory, no revolutionary ideas have surfaced in this area for several decades. After the work of Merton (1969, 1971) carved out in stone the notion that, within certain assumptions, long–term, multiperiod, investment solutions are identical to the single period ones; the wealth management industry has used the myopic mean-variance portfolio optimization as the cornerstone of the long term asset allocation process. However, new developments over the last decade have brought back in focus the problem of long term asset allocation.

The main assumption underlying the Markowitz optimal portfolio construction procedure is that the variance of the return distribution is all that is needed in order to characterize the risk of the investment. A whole new field of interest has emerged with the introduction of the notion of Value at Risk (VAR) by RiskMetricsTM (1996), filling a large gap in the financial industry’s need to assess the extreme risks associated with most investments. The whole idea behind the use of VAR is to encapsulate into a single number all the information about the possible portfolio losses implied by the left hand side tail of the return distribution, especially in the case when this distribution is not normal. VAR has quickly become the universally accepted measure of risk for short term risk, for a review see Duffie and Pan (1997). The use of VAR has become the de facto norm in risk management, such that the controlling authorities have imposed regulatory constraints on the asset allocations of financial institutions based on the estimation of VAR. It was natural to attempt to use the VAR as the measure of risk that has to balance against expected returns for the purpose of portfolio optimization. However, Artzner (1997, 1999) laid out the desirable mathematical properties that a risk measure has to possess in order to reflect the commonly accepted behavior of rational investors. Artzner et al. have defined the conditions for the so called “coherent risk measures”, and unfortunately VAR does not satisfy all of them. What VAR lacks is the property of convexity; therefore, it does not provide the investors with an incentive to diversify their investment.
As an alternative to VAR, a new risk measure has recently emerged: the Conditional Value at Risk (CVaR). The CVaR extends the notion of VAR, it is a more conservative measure of risk than VAR (CVaR is always larger than VAR), and fortunately, it is also a coherent risk measure in the sense defined by Artzner et al. This measure has been utilized for the first time for the purpose of portfolio optimization by Rockafellar and Uryasev (2000, 2002) in conjunction with a linear programming algorithm. This combination is particularly attractive since it allows an easy implementation using standard linear programming packages. As a consequence, CVaR has gained increasing acceptance in the academic community as the method of choice for portfolio optimizations that need to go beyond the simple mean-variance framework. Notable examples are portfolio optimizations using derivatives by Alexander et al. (2004) and structured products by Martellini et al. (2005). From academic research literature, the CVaR optimization method has started to trickle towards practical applications in the financial industry. Very recently, a portfolio optimization software package called CVaR eXpert (2005) has been released to the market.

Another important development in recent years is the tremendous growth of the computing power available for numerical calculations. Numerical approaches to portfolio optimization have become a common sighting in literature. Numerical approaches to multiperiod portfolio optimization have also been developed. Of particular importance to the present work is the paper of Bogentoft et al. (2001) on Asset/Liability management of pension funds which uses the CVaR optimization approach in a multiperiod setting.

In the last years, it has become more and more evident that the myopic allocation predicted by Merton’s model is too simple (see, e.g., Campbell and Viceira, 2002). The approximation that the capital market assumptions are constant in time is too limiting. The need of investors to hedge against the change in investment opportunities leads to changes in the long-term asset allocations. In addition, the fact that the needs and opportunities for long term investors are quite different from those of concern over short term became more and more obvious. For example, cash is a riskless asset over the short term, but it carries a large risk due to inflation over the long term. Meanwhile, industry has developed its own ways of dealing with the problems relating to long-term asset allocation, often totally disconnected from the results of academic research. A notable example is the conventional wisdom that younger investors should invest a larger proportion to equity and less in bonds, as opposed to an older one, who should allocate more to bonds and less to equity. Another example is the so
called “asset allocation puzzle” relating to the fact that investment advisors usually recommend different proportions for the risky assets in a portfolio according to the risk profile of the investor.

In the context of the revival of the portfolio optimization research, the present work has the goal of exploring yet another facet of the long term investment problem. It aims to describe the relationship between the goal of a good risk-gain balance over the long term and the short-term risk aversion of most investors. The choice of CVaR as an optimization method is based on its flexibility and on the simplicity of its implementation using a standard linear programming package. I start this study by solving a single-period optimization problem using CVaR. I present results that show the dependence of the efficient frontier on the investment horizon when using a buy-and-hold strategy. The central point of this work is the construction of constrained long-term efficient frontiers. The constraints correspond to short term probability distributions of returns that are based on market assumptions which differ from long-term ones. The method is quite general and could be extended to a truly multiperiod problem. However, the present two-period form (short and long term) is best suited for combining what the industry calls strategic and tactical asset allocations. The use of these industry coined terms is not accidental, since the practical application of this method for the needs of private banking industry was the main driver of this work.

The organization of this thesis work is as follows. Section 2 presents a few theoretical formulas of the portfolio optimization problem. I briefly describe the mean-variance approach, and then I give an overview of VAR and CVaR and of their main mathematical properties. In the end I present the equations for CVaR portfolio optimization using linear programming which are the base for the MATLAB implementation. Section 3 shows single term portfolio optimization results for different investment horizons. The distributions used are either historical or simulated using a Monte Carlo technique based on the Skewed Student-T distribution. The next section describes the multiperiod constrained portfolio optimization. I give a short overview of the formulation of the problem, then describe how I chose the different market assumptions for short and long term horizons. Finally, I demonstrate results obtained for one constrained optimization. The last section summarizes my conclusions.
2. Portfolio Optimization Methods

2.1 Mean-Variance Portfolio Optimization

Markowitz (1956) portfolio theory has been a cornerstone of the wealth management process during most of the last half-century. Asset allocation based on mean-variance optimization represents the benchmark against which any new allocation technique has to be measured. I give here a brief overview of the key formulas leading to the construction of the Markowitz efficient frontier following loosely the notations of Vanini and Vignola (2001).

Assume that we have $N$ assets and that the common return distribution at the end of the investment period $t$ is described by the mean vector, $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, $\{\mu_i\}_{i=1,N}$, and by the covariance matrix, $V_{i,j}$. For an arbitrary portfolio, $\overline{w} = (w_1, w_2, \ldots, w_N) \in \mathbb{R}^N$, $\left( \sum w_i = 1 \right)$, the return, $R(\overline{w})$, and variance, $Var(\overline{w})$, can be calculated as:

$$R(\overline{w}) = \sum_{i=1,N} w_i \mu_i = \langle \overline{w}, \mu \rangle,$$

$$Var(\overline{w}) = \sum_{i=1,N} \sum_{j=1,N} w_i V_{i,j} w_j = \langle \overline{w}, \overline{V} \overline{w} \rangle.$$

Mean-variance portfolio optimization means finding that portfolio allocation, $\overline{w}^*$, which for a fixed level of the expected return will have the minimum possible value of $Var(\overline{w})$. Additional constraints on $\overline{w}$ may be imposed, for example, disallowing short selling ($0 \leq w_i$) or setting a hard limit to the maximum allocation on some assets. Using the notation:

$$X = \left\{ \overline{w} \in \mathbb{R}^N \mid w_i \geq 0, \sum_{i=1,N} w_i = 1, \sum_{i=1,N} w_i \cdot \mu_i \geq R_0 \right\},$$
to specify the set of allowed values for the portfolio weights, the optimization problem can be simply stated as:

$$\min_{\mathbf{w}} \{ Var(\mathbf{w}) \} , \quad \mathbf{w} \in X .$$

Using the notations of Vanini and Vignola (2001), the solution to this problem can be written as follows:

$$\mathbf{w}^* = R_0 \mathbf{w}_0 + \mathbf{w}_1 ,$$

with

$$\mathbf{w}_0 = \frac{1}{\Delta} \left( < e, V^{-1} e > V^{-1} \mathbf{\mu} - < e, V^{-1} \mathbf{\mu} > V^{-1} e \right) ,$$

$$\mathbf{w}_1 = \frac{1}{\Delta} \left( < e, V^{-1} \mathbf{\mu} > V^{-1} \mathbf{\mu} - < \mathbf{\mu}, V^{-1} \mathbf{\mu} > V^{-1} e \right) ,$$

$$\Delta = \| \sigma^{-1} e \|^2 \| \sigma^{-1} \mathbf{\mu} \|^2 - \left[ < \sigma^{-1} e, \sigma^{-1} \mathbf{\mu} > \right]^2 ,$$

$$\| x \| = \sqrt{< x, x >} , \quad e = (1,1,...1).$$

This solution leads to the construction of the well known mean-variance efficient frontier.

**Figure 1** The mean-variance efficient frontier.
Figure 1 illustrates the construction of the Markowitz efficient frontier using the frame of expected return vs. standard deviation (instead of variance). Note that the frontier extends to the left only up to the minimum variance portfolio. It is evident from the above formulas that the mean-variance optimization only takes into account the first two moments of the return distribution of the considered assets. This treatment is exact in the case of normally distributed returns.

The riskless asset is not included among the $N$ assets considered above. Allocation between a riskless asset and the market portfolio is performed in the second step, along the tangent to the efficient frontier that goes through the point representing the riskless asset. This is where the risk aversion of the investor comes into play; more risk-averse investors would choose portfolios located on the tangent closer to the riskless asset, while investors with higher risk tolerance would choose portfolios closer to the tangent portfolio. The composition of the risky asset part is, however, the same for all investors, regardless of their risk aversion.
2.2 Value at Risk and Conditional Value at Risk

2.2.1 The Value at Risk

The Value at Risk (VAR) is a statistical measure of the possible portfolio loss over a predefined period. Since the notion of Conditional Value at Risk (CVaR) is closely related to that of VAR, I start by presenting its definition and a short description of some of its properties. The following description is based on the lecture notes on “Optimal Portfolio Construction” of Vanini and Vignola (2001), alternatively, Duffie and Pan (1997) give an overview of VaR. The definition of VAR is related to the probability distribution of the portfolio returns at the end of the investment period \( t \). Let the portfolio change in value over this interval be:

\[
\Delta V^t = V^t - V^0.
\]

If one chooses a confidence level \( \alpha \), then the value at risk with confidence \( \alpha \) and over the time period ending at \( t \), \( VAR^t_\alpha \), is defined by:

\[
P(\Delta V^t < -VAR^t_\alpha) = 1 - \alpha.
\]

Here \( P(a) \) is the probability of occurrence for the event \( a \). In risk management applications, \( \alpha \) has usually a value of 95% or 99%. It can be easily seen from the above formula that VAR is nothing else than the maximum portfolio loss incurred if we exclude the worst \( 1 - \alpha \) cases. Although, the VAR is usually defined in terms of a dollar value, we can also see it in terms of absolute returns for our portfolio. If

\[
R^t = V^t / V^0,
\]

is the total return on the portfolio over period \( t \), then one defines \( VAR^t_\alpha \) by the equation:

\[
P(R^t < -VAR^t_\alpha) = 1 - \alpha.
\]
The VAR is determined by the extreme left hand side of the portfolio return distribution. If \( f(R) \) is the probability distribution function (pdf) of the portfolio returns, and \( F(R) \) is the associated cumulative distribution function (cdf), then the VAR can be seen as the \( 1-\alpha \) percentile of this distribution:

\[
VAR^\alpha_t = F^{-1}(\alpha).
\]

If the portfolio returns are normally distributed, VAR can be easily calculated using the percentiles of the normal distribution. Usually, the price processes of the portfolio is assumed to follow a geometric Brownian motion stochastic process:

\[
dV_t = V_t (\mu \cdot dt + \sigma \cdot dW_t),
\]

where \( dW_t \) is a standard Brownian motion. Then the absolute returns are distributed according to a shifted log-normal distribution:

\[
R^t = \exp(r^t) - 1; \quad r^t \sim N(\mu t - \frac{\sigma^2}{2} t, \sigma^2 t),
\]

where \( r^t \) are the continuously compounded returns which follow a normal distribution. For time intervals that are quite short (1, 5, or 20 days) the returns \( r^t \) are small and a Taylor series expansion of the exponential, up to the linear term, leaves just:

\[
R^t \approx r^t \sim N(\mu t - \frac{\sigma^2}{2} t, \sigma^2 t).
\]

Also, assuming that \((\mu - \frac{\sigma^2}{2})t \approx 0\), the value at risk can be simply calculated as:

\[
VAR^\alpha_t = V_t z_{1-\alpha} \sqrt{t},
\]
where \( z_{1-\alpha} \) is the \( 1-\alpha \) percentile of the standard normal distribution. This percentile has the value 1.65 for \( 1-\alpha = 5\% \) and 2.66 for \( 1-\alpha = 1\% \). Notice that this expression depends only on the standard deviation of the portfolio and therefore, as in the Markowitz approach, the full measure of the portfolio risk is given by its standard deviation. This expression is the most basic measure of risk which is defined by RiskMetricsTM (1996), and is used in many risk management applications.

In the context of portfolio optimization, I introduce first VAR, and further CVaR, as measures of risk. Risk is a characteristic of the portfolio return distribution that is disliked by any rational investor. In order to be able to evaluate the efficiency of VAR and CVaR as measures of risk, I need to use the notion of stochastic dominance refering to the return distributions. Taking two random variables \( Y_1 \) and \( Y_2 \) (think of them as the negative of absolute returns, i.e., the losses, of two portfolios), one can define the following binary relations:

\[ a) \text{ Stochastic dominance of order 1, } Y_1 \prec_{sd1} Y_2, \text{ which is true if } \]
\[ E[f(Y_1)] \leq E[f(Y_2)] \]
\[ \text{for all functions } f \text{ that are integrable and monotonic.} \]

\[ b) \text{ Stochastic dominance of order 2, } Y_1 \prec_{sd2} Y_2, \text{ which is true if } \]
\[ E[f(Y_1)] \leq E[f(Y_2)] \]
\[ \text{for all functions } f \text{ that are integrable, monotonic, and concave.} \]

\[ c) \text{ Monotonic dominance of order 1, } Y_1 \prec_{md2} Y_2, \text{ which is true if } \]
\[ E[f(Y_1)] \leq E[f(Y_2)] \]
\[ \text{for all functions } f \text{ that are integrable and concave.} \]

The first relation, \( sd1 \), being true is equivalent to saying that the portfolio corresponding to \( Y_1 \) assigns higher payoffs than that corresponding to \( Y_2 \), in any possible state. In this case any investor who prefers more to less should prefer the portfolio described by \( Y_1 \). However, this type of relationship can not be established between any pair of distributions, consequence of the fact that \( sd1 \) does not take into account the risk aversion.
This is done by \( \text{sd}2 \) which accounts for the concavity of the utility function. It can be shown that the following statements are true about the stochastic dominance relations defined above.

1. \( Y_1 \prec_{\text{sd}1} Y_2 \Rightarrow Y_1 \prec_{\text{sd}2} Y_2 \),

2. \( Y_1 \prec_{\text{sd}2} Y_2 \Leftrightarrow (Y_1 \prec_{\text{sd}1} Y_2 \text{ } \& \text{ } Y_1 \prec_{\text{sd}1} Y_2) \),

3. \( Y_1 \prec_{\text{sd}2} Y_2 \Leftrightarrow \int_{-\infty}^{x} F_{Y_1}(u)du \leq \int_{-\infty}^{x} F_{Y_2}(u)du \text{ for all } x. \)

For a proof of these relations see Vanini (2001).

It is important to mention that \( \text{VAR} \) has some properties which make it a desirable candidate as a risk measure. The following properties refer to the value at risk, \( \text{VAR}_\alpha(Y) \), which is defined for portfolios described by \( Y \) and for a confidence level \( \alpha \):

1. \( \text{VAR}_\alpha(Y + c) = \text{VAR}_\alpha(Y) + c \) for any real number \( c \),

2. \( \text{VAR}_\alpha(c \cdot Y) = c \cdot \text{VAR}_\alpha(Y) \) for any real number \( c > 0 \),

3. \( \text{VAR}_\alpha(Y) = -\text{VAR}_{1-\alpha}(-Y) \),

4. \( Y_1 \prec_{\text{sd}1} Y_2 \Rightarrow \text{VAR}_\alpha(Y_1) \leq \text{VAR}_\alpha(Y_2) \).

These properties are almost all that is necessary to make \( \text{VAR} \) a coherent risk measure in the sense of Artzner (1997, 1999). However, the main drawback of \( \text{VAR} \) as a portfolio risk measure is that it lacks the convexity property. More explicitly, it can happen that a portfolio which contains two assets is riskier, according to \( \text{VAR} \), than any of the two composing assets. For an example of such a case see Artzner (1999). The consequence of this is that anyone who uses \( \text{VAR} \) as a measure of portfolio risk will generally not have the incentive to diversify the portfolio.
2.2.2 The Conditional Value at Risk

The Conditional Value at Risk (CVaR) is closely related to the VAR; however, it has mathematical properties which allow it to be a better measure of portfolio risk for the purpose of portfolio optimization as shown by Rockafellar and Uryasev (2000, 2001, 2002). The following description of its key properties, which relate to portfolio selection, closely follows the notes of Vanini and Vignola (2001). Using the same meaning as above for the random variable $Y$ (the portfolio loss over the considered investment period), one can define the conditional value at risk, $CVaR_\alpha(Y)$ as the solution of the minimization problem:

$$CVaR_\alpha(Y) = \inf \left\{ a + \frac{1}{1-\alpha} E[Y-a]^+ : a \in \mathbb{R} \right\}.$$

Although this definition of CVaR is not the most common one, it is the one that shows explicitly one of its key properties: that CVaR is the solution of a linear minimization problem. It can be shown that the value of $a$ that minimizes the above expression is nothing else than $VAR_\alpha(Y)$, the usual value at risk, as it is demonstrated by Rockafellar and Uryasev (2000). Another key result shown by Rockafellar and Uryasev (2000) is that the above definition of CVaR is equivalent to the following one:

$$CVaR_\alpha(Y) = E[Y | Y > VAR_\alpha(Y)].$$

This way of writing CVaR explains its name, since it involves a conditional expectation. CVaR is nothing else than the expectation of the portfolio loss, when the expectation is calculated conditional on the loss exceeding the $\alpha$ - VAR of the portfolio. Figure 1, extracted from Vanini (2001), illustrates the relation between VAR and CVaR.

It is clear that VAR only measures the total probability that is covered by the left hand side of the pdf of the portfolio returns (that is the r.h.s. of the loss distribution). However, VAR is totally insensitive to modification of the shape of the pdf to the left of the $\alpha$ - percentile. Meanwhile, CVaR goes one step further in describing the tail of the distribution; it measures the first moment of the piece of the distribution that is beyond the VAR. This is by
no means a complete description of the portfolio risk, but it is certainly an improvement over the use of VAR.

![Figure 2 VaR and CVaR for possible losses of a portfolio. Reproduction after Vanini and Vignola (2001).](image)

The fact that CVaR is a better measure of risk is also shown by the following properties:

1. $CVaR_\alpha(Y + c) = CVaR_\alpha(Y) + c$ for any real number $c$,

2. $CVaR_\alpha(c \cdot Y) = c \cdot CVaR_\alpha(Y)$ for any real number $c > 0$,

3. $E[Y] = (1 - \alpha) \cdot CVaR_\alpha(Y) - \alpha \cdot CVaR_{1-\alpha}(-Y),$

4. $Y_1 \overset{\text{ind}}{\sim} Y_2 \Rightarrow CVaR_\alpha(Y_1) \leq CVaR_\alpha(Y_2),$

5. $Y_1 \overset{\text{sd}}{\sim} Y_2 \Rightarrow CVaR_\alpha(Y_1) \leq CVaR_\alpha(Y_2),$

6. $CVaR_\alpha(\lambda Y_1 + (1 - \lambda)Y_2) \leq \lambda CVaR_\alpha(Y_1) + (1 - \lambda)CVaR_\alpha(Y_2),$
Leading trails towards the proofs of these properties can be found in Vanini and Vignola (2001). The most important change compared to the similar properties of VAR is the addition of property number 6, convexity. Based on the above properties, one can insert CVaR in the category of coherent risk measures, in the sense of Artzner (1999). This has important consequences when CVaR is used as a minimization criterion in portfolio optimization applications. Convexity guarantees that there are no local minima; therefore, it guarantees that if a minimum CVaR portfolio can be found, then this portfolio is optimal.

Comparison of VAR and CVaR can also be based on the following statements:

1. \( CVaR_\alpha(Y) \leq VAR_\alpha(Y), \)

2. for non-negative \( Y \)
\[
\left( \frac{E[Y^n] - (1 - \alpha) CVaR_\alpha(Y^n)}{\alpha} \right)^{1/n} \rightarrow VAR_\alpha(Y), \quad \text{for } n \rightarrow \infty
\]

Most notably, the first of the two states that CVaR is a more conservative risk measure than VAR. Therefore, an investor optimizing for minimum CVaR would take at most the same risk as if the optimization would be done using VAR.
2.3 Portfolio Construction using CVaR Minimization

As already mentioned, CVaR is a risk measure that is well suited for portfolio optimization problems. On the one hand, it can account for the non-normality of asset returns; therefore, it brings an improvement over the mean-variance optimization method. On the other hand, CVaR has better mathematical properties compared to VAR, namely, it is a coherent measure of risk. Due to these advantages it has been chosen by several authors, starting with Rockafellar and Uryasev (2000, 2002) as a measure of risk in a mean-CVaR optimization algorithm that improves over the mean-variance optimization method of Markowitz (1956). Below is a description of this optimization method.

Consider $N$ assets that have a common probability density function $f(y)$. For an arbitrary portfolio $\overline{w} = (w_1, w_2, ..., w_N) \in \mathbb{R}^N$, $\left( \sum w_i = 1 \right)$, I denote by $g(\overline{w}, y)$ the portfolio loss function associated with the state $y$. Then the CVaR associated with the returns/loss distribution of portfolio $\overline{w}$ is:

$$CVaR_{\alpha}(\overline{w}) = \frac{1}{1 - \alpha} \int_{g(\overline{w}, y) \leq VaR_{\alpha}(\overline{w})} g(\overline{w}, y) f(y) dy$$

Mean-CVaR portfolio optimization means nothing else than finding that portfolio allocation which for a fixed level of expected return will have the minimum possible value of $CVaR_{\alpha}(\overline{w})$ while also satisfying additional requirements (for example no short-selling and no leverage: $0 \leq w_i \leq 1$, $i = 1, .., N$). In fact, it can be shown that there are two more equivalent ways of formulating the optimization problem: i) fix the CVaR and maximize the expected return, ii) minimize a linear combination of the CVaR and the negative of the expected return; for details see Vanini (2001). If $R(\overline{w}, f)$ is the expected return of the portfolio $\overline{w}$ and if $\Omega \subset \mathbb{R}^N$ is the set of the allowed portfolio weights compatible with the required constraints, then for the following three problems:
there is a one to one relationship between the parameters \( \lambda, R_0 \), and \( CVaR_0 \) such that the three problems yield the same efficient frontier. In all three formulations, the CVaR is difficult to evaluate if we use the above formula that depends on VAR. However, according to Rockafellar (2000) one can use the following alternative approach. Consider the function:

\[
H_a(a, \overline{w}) = a - \frac{1}{1 - \alpha} \int \left[ g(\overline{w}, y) - a \right]^+ f(y) dy.
\]

As stated above, the minimization of \( H_a(a, \overline{w}) \) with respect to \( a \) yields as value of \( H \) the CVaR of the portfolio \( \overline{w} \), while the minimizing \( a \) is nothing else than the VAR of this portfolio. A more powerful statement of Uryasev is that the solution of the minimization problem

\[
\min_{a, \pi} \{ H_a(a, \overline{w}) \}, \quad R(\overline{w}; f) \geq R_0, \quad a \in \mathbb{R}, \ \overline{w} \in \Omega,
\]

gives, in the same time, the CVaR optimal portfolio, as well as the CVaR and the VAR of the optimal portfolio. The strength of this approach lies in the fact that \( H_a(a, \overline{w}) \) is a convex function and is also a linear function both in \( a \) and in the portfolio weights. Therefore, the portfolio optimization problem is reduced to a convex linear optimization problem of dimension \( N+1 \).

It is interesting to consider the optimization problem in the case where the asset returns are normally distributed, i.e., \( f(y) \) is a multivariate normal distribution. Let \( \{ \mu_i \}_{i=1}^{N} \) be the expected returns for the \( N \) assets calculated using \( f(y) \). The optimization problem can be defined easily with the help of the set \( X \) of the admissible portfolios that have a fixed expected return \( R_0 \):
\[ X = \left\{ \bar{w} \in \mathbb{R}^N \mid w_i \geq 0, \sum_{i=1}^N w_i = 1, \sum_{i=1}^N w_i \cdot \mu_i \geq R_0, \right\}. \]

It has been shown by Rockafellar and Uryasev (2000) that the three optimizations

1. \( \min_{\pi} \{ CVaR_\alpha(\bar{w}) \}, \quad \bar{w} \in X, \)
2. \( \min_{\pi} \{ VaR_\alpha(\bar{w}) \}, \quad \bar{w} \in X, \)
3. \( \min_{\pi} \{ Var(\bar{w}) \}, \quad \bar{w} \in X, \)

yield the same optimal portfolio. The third optimization is nothing else than the usual mean-variance optimization of Markowitz. In fact, one can express both the VAR and the CVaR of the portfolios belonging to \( X \) as:

\[
CVaR_\alpha(\bar{w}) = R_0 + c_1(\alpha) \sqrt{Var(\bar{w})}, \quad \bar{w} \in X,
\]

\[
VaR_\alpha(\bar{w}) = R_0 + c_2(\alpha) \sqrt{Var(\bar{w})}, \quad \bar{w} \in X.
\]

where \( c_{1,2} \) are functions of \( \alpha \) only. It is clear that the portfolio which has the minimum variance will also have the minimum VAR and CVaR. This statement is of particular interest because it allows one to define a benchmark case on which to test the CVaR optimization. A similar result has been obtained in the case of elliptic distributions by Embrechts et al. (2002).

Up to this point, the assets which compose the investment universe have been specified through the distribution function of their returns \( f(y) \). Using an explicit expression for \( f(y) \), one can attempt to solve the above linear optimization problem analytically. Most of the time, however, the distribution \( f(y) \) is not available in closed form or, if it is, the problem is too complicated to be solved in closed form. A numerical solution is then preferred. A set of scenarios are generated such that they sample the space of possible returns.
according to the distribution $f(y)$. Let $\{y_j\}_{j=1}^J$ be a set of $J$ such scenarios. Then the integral in function $H_\alpha(a, \bar{w})$ can be approximated as:

$$H_\alpha(a, \bar{w}) = a - \frac{1}{1 - \alpha} \sum_{j=1}^J [g(\bar{w}, y_j) - a]^+. $$

With the help of a set of $J$ dummy variables $\{z_j\}_{j=1}^J$, the problem of minimizing $H_\alpha(a, \bar{w})$ can be reduced to a standard linear programming problem:

$$\min_\pi \left\{ a - \frac{1}{1 - \alpha} \sum_{j=1}^J z_j \right\}$$

s.t

\begin{align*}
z_j & \geq g(\bar{w}, y_j) - a, \\
z_j & \geq 0, \\
R(\bar{w}, f) & \geq R_0, \\
a & \in \mathbb{R}, \; \bar{w} & \in \Omega.
\end{align*}

More explicitly, assume that in scenario $y_j$ return of the asset $i$ is $R_i^j$, then the portfolio loss and expected return can be expressed as:

\begin{align*}
g(\bar{w}, y_j) &= -w_i \cdot R_i^j, \\
R(\bar{w}, f) &= \sum_{i=1}^N w_i \cdot E[R_i].
\end{align*}

For long only portfolios, one obtains in the end the following linear programming problem that can be solved numerically:

$$\min_{a, \pi} \left\{ a - \frac{1}{1 - \alpha} \sum_{j=1}^J z_j \right\}$$

s.t

\begin{align*}
z_j & \geq 0, \\
z_j & \geq -w_i \cdot R_i^j - a, \\
w_i & \geq 0, \\
\sum_{i=1}^N w_i \cdot E[R_i] & \geq R_0, \\
\sum_{i=1}^N w_i &= 1.
\end{align*}
The size of the linear programming problem to be solved is $J + N + 1$.

The main limitation to the accuracy of the CVaR optimization method is that a finite sampling can give only an approximated description of the assumed distribution of the future asset returns. Therefore, the accuracy of the CVaR optimization is essentially limited by the size of the sampling of the asset distribution. On its turn, the size of the sampling can be limited by two factors: the size of the available time series, when using historical data, and by the size of the linear programming problem that can be solved numerically given the available hardware/software. Due to these reasons the examples in this work involve a few hundred scenarios at most.

A standard linear programming subroutine like that included in the Optimization Toolbox of MATLAB V7.0.0(R14) can solve the above problems for sizes of up to a few thousands. However, taking advantage of the fact that the coefficient matrix for the constraints is very sparse, optimizations over more than 20000 scenarios have been reported using the CPLEX (Rockafellar, 2000) or MOSEK optimization packages (Alexander, 2004).

For this work I have used the `linprog()` linear programming function which is part of the Optimization Toolbox of MATLAB 7.0. Problem of sizes ranging from 500 to 1000 scenarios can be solved in a few minutes on 1.8 GHz AMD Athlon XP with 512 MB RAM. In the Appendix, I present the MATLAB functions which construct the matrix form of the above optimization problem and call the library routine for solving it. The problem formulations that I implemented and are listed in the Appendix are: i) CVaR minimization for an imposed minimum return; ii) Expected return maximization for an imposed maximum CVaR.
3. Single Period Portfolio Optimizations

3.1 Optimization using Modified Historic Return Distributions

Portfolio optimizations are usually based on specific assumptions about the nature and parameters of the distribution of future asset returns. Most of the time, the type of the distribution used is imposed by the limitations of the portfolio optimization method. This is the case of the mean-variance optimization which assumes normally distributed returns. One advantage of the CVaR optimization method is that it can be applied to an arbitrary sample of predicted returns, since it does not involve specific assumptions on the return distributions of the assets considered.

The number of parameters that fully describe the returns of a set of assets for the purpose of Markowitz optimization is relatively small. All that is needed is a view on the values of expected returns and on the Variance-Covariance matrix of the investment instruments that are considered. Values of these parameters are usually inferred by analysts by putting together a lot of information that ranges from historical data and up to predictions and expectations regarding the behavior of the economy. For the purpose of long-term (strategic) asset allocation it is customary to use predictions of mean and variance based on long term expectations on the economy (5-20 years) and historical data for the correlation of different assets.

As a first example of CVaR-based portfolio optimization I have chosen four asset classes: equity, US government bonds, US high-yield bonds, and emerging market bonds. While the first two classes are standard components of any portfolio allocation, the last two, and particularly the emerging market bonds, provide examples of historical time series that show a significant departure from normality. The simplest assumption that can be made about the future distribution of these returns (but also a very naïve one), is that it fully resembles the historical one. In this case the full set of historical returns can be used as the scenario set over which the CVaR optimization program is performed. This is first the example that I have used to test the CVaR optimization method, however, with a slight modification of the historical time series as explained below.
Very detailed predictions about the expected returns and covariance matrices are available in most investment department of financial institutions. However, there is little information about the higher order moments of the distributions (e.g., skewness, kurtosis) since mean-variance optimization is insensitive to the value of these parameters. For the example that follows I have taken the historical distributions, including historical variance-covariance and also higher moments, but I have shifted the means of the distributions to be in agreement with the figures and methodology of making long term predictions used by Merrill Lynch analysts (Merrill Lynch, 2005).

The historical time series that I used contain monthly data on four indices, representing the four asset classes, from January 1994 up to March 2005. The four market indices that I have used are:

- MSCI: MSCI World Index TR (% Total Return),
- USGv: Citigroup U.S. 1+ Yr Gvt TR (% Total Return),
- HYB: ML Hi-Yld Master (Cash Pay Only) (% Total Return),
- EMB: JPM EMBI+ Composite TR (% Total Return).

The first four moments of the series, that have been modified such their means are in agreement with our expectations, are presented in Table 1 along with their correlation coefficients.

I compare results obtained using, on the one hand, mean-variance optimization, and on the other hand, mean-CVaR optimization. Using either method, I obtain the portfolio compositions along the corresponding efficient frontier. In general, for non-normal distributions one obtains two different efficient frontiers involving different portfolio compositions. Figure 3 shows the results for the optimizations using the assumptions presented in Table 1 and a CVaR confidence level $\alpha = 95\%$. Both efficient frontiers are plotted in either of the two coordinate frames: expected return vs. standard deviation, and expected return vs. CVaR. Note the difference between the two efficient frontiers and also the fact that each of them looks optimal in its own frame. In a graphical representation of return vs. risk, optimal portfolios can be found to the left, or above, for the same expected return, or level of risk, respectively. The differences seen in Figure 3 are due to the departure from normality of the high yield bonds and, especially, of the emerging market bonds, which
present a significant negative skewness. The portfolio compositions corresponding to the two efficient frontiers are shown in Figure 4. CVaR optimization reduces the share of high yield bonds and also, to a certain extent, that of the emerging market bonds.

**Table 1.** Statistical properties of the series with 1 month frequency used for portfolio optimization. The series are historical monthly data with adjusted mean. Sample size: 135.

<table>
<thead>
<tr>
<th>Statistical Properties</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0076</td>
<td>0.0037</td>
<td>0.0053</td>
<td>0.0076</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0409</td>
<td>0.0136</td>
<td>0.0189</td>
<td>0.0485</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.5849</td>
<td>-0.4833</td>
<td>-0.7376</td>
<td>-1.9691</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.5308</td>
<td>3.8703</td>
<td>6.3378</td>
<td>12.5568</td>
</tr>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>4.50%</td>
<td>6.50%</td>
<td>9.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>15.47%</td>
<td>4.90%</td>
<td>6.95%</td>
<td>18.37%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI</td>
<td>1.0000</td>
<td>-0.1479</td>
<td>0.5020</td>
<td>0.5167</td>
</tr>
<tr>
<td>USGv</td>
<td>-0.1479</td>
<td>1.0000</td>
<td>0.1165</td>
<td>0.1460</td>
</tr>
<tr>
<td>HYB</td>
<td>0.5020</td>
<td>0.1165</td>
<td>1.0000</td>
<td>0.4587</td>
</tr>
<tr>
<td>EMB</td>
<td>0.5167</td>
<td>0.1460</td>
<td>0.4587</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Figures 5 and 6 display the same efficient frontiers as Figures 3 and 4, but using a confidence level $\alpha = 90\%$ for CVaR calculation. Decreasing $\alpha$, clearly brings the two frontiers much closer to each other. A high $\alpha$ means that only the extreme left part of the return distribution contributes to CVaR calculation and this is where the differences between normal and non-normal distributions are most visible. When $\alpha$ decreases a larger portion of the distribution is involved in the calculation of CVaR and the portion where the departure from normality is largest, the extreme tail, has a lower influence on the final result. Differences that are still visible for $\alpha = 90\%$ are most likely due to statistical errors, consequence of the rather poor statistics used in this optimization: between 100 and 150 points for every optimization.
Figure 3 Efficient frontiers obtained using mean-variance (green) and mean-CVaR (blue) optimization. Top graph shows both efficient frontiers as returns vs. standard deviation and bottom graph shows the same efficient frontiers as returns vs. CVaR. Optimization is performed on historical monthly series with adjusted means. CVaR is computed using $\alpha = 0.95$. 
Figure 4  Asset allocation for the efficient frontiers plotted in Figure 3. Portfolios are numbered for each frontier from left (least risky) to right (most risky). Note that portfolios with the same number obtained using different optimization methods may correspond to different Return and/or Risk.
Figure 5 Same as Figure 3 but CVaR is computed using $\alpha = 0.90$. 
Figure 6 Same as Figure 4 but CVaR is computed using $\alpha = 0.90$. 
The most common application of strategic asset allocation is in the case of portfolios that are rebalanced periodically. In this case, the assumptions of Merton (1971, 1973) lead to a multi-period optimal allocation solution that is independent of the investment horizon. However, for a buy-and-hold investor, the allocation will depend on the market assumptions that are compatible with his investment horizon. I investigate here the consequences of such an investment philosophy of a long term buy-and-hold investor.

As examples of long term, buy-and-hold optimizations, I investigate the cases of 12 month and 36 month investment periods. In order to sample the asset distributions, I use as a starting point the same monthly distributions as before. However, I compound 12, respectively 36, monthly returns on a rolling window that moves along the whole 14-year period of the initial series. Note that the resulting series contains returns of 1 or 3 year overlapping periods. The obtained series are again shifted such that the annualized average return in each case is in agreement with the long term forecasts.

Table 2 Statistical properties of the series with 12 month frequency obtained by compounding consecutive monthly returns in a rolling window. Sample size: 124.

<table>
<thead>
<tr>
<th>Statistical Properties</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0950</td>
<td>0.0450</td>
<td>0.0650</td>
<td>0.0950</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.1697</td>
<td>0.0479</td>
<td>0.0825</td>
<td>0.1731</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.5179</td>
<td>-0.1120</td>
<td>0.4651</td>
<td>-0.2166</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.4252</td>
<td>2.4525</td>
<td>2.8271</td>
<td>2.5256</td>
</tr>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>4.50%</td>
<td>6.50%</td>
<td>9.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>16.97%</td>
<td>4.79%</td>
<td>8.25%</td>
<td>17.31%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI</td>
<td>1.0000</td>
<td>-0.3768</td>
<td>0.5790</td>
<td>0.3243</td>
</tr>
<tr>
<td>USGv</td>
<td>-0.3768</td>
<td>1.0000</td>
<td>0.0439</td>
<td>-0.0867</td>
</tr>
<tr>
<td>HYB</td>
<td>0.5790</td>
<td>0.0439</td>
<td>1.0000</td>
<td>0.4423</td>
</tr>
<tr>
<td>EMB</td>
<td>0.3243</td>
<td>-0.0867</td>
<td>0.4423</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Table 3 Statistical properties of the series with 36 month frequency obtained by compounding consecutive monthly returns in a rolling window. Sample size: 100.

<table>
<thead>
<tr>
<th>Statistical Properties</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.3129</td>
<td>0.1412</td>
<td>0.2080</td>
<td>0.3129</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.3833</td>
<td>0.0522</td>
<td>0.1586</td>
<td>0.2659</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.2644</td>
<td>0.6386</td>
<td>0.0063</td>
<td>1.2704</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>1.6727</td>
<td>2.5054</td>
<td>1.7952</td>
<td>4.0818</td>
</tr>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>4.50%</td>
<td>6.50%</td>
<td>9.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>18.21%</td>
<td>2.76%</td>
<td>8.05%</td>
<td>12.72%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>EMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI</td>
<td>1.0000</td>
<td>-0.2337</td>
<td>0.5541</td>
<td>0.2427</td>
</tr>
<tr>
<td>USGv</td>
<td>-0.2337</td>
<td>1.0000</td>
<td>0.0321</td>
<td>0.2390</td>
</tr>
<tr>
<td>HYB</td>
<td>0.5541</td>
<td>0.0321</td>
<td>1.0000</td>
<td>0.6634</td>
</tr>
<tr>
<td>EMB</td>
<td>0.2427</td>
<td>0.2390</td>
<td>0.6634</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The statistical properties of the resulting series are shown in Tables 2 and 3, respectively, for the 12 and 36 month investment horizons. Note that compounding of 12 or 36 consecutive monthly distributions brings the resulting distributions significantly closer to normality. As a result, the mean-variance and the mean-CVaR efficient frontiers are much closer to each other, when regarding the return vs. risk plots in Figures 7 and 9, as well as when looking at asset allocations along the efficient frontiers in Figures 8 and 10.

The most striking difference can be observed for the 36 month investment where the mean-CVaR efficient frontier extends less to the left, compared to the Markowitz efficient frontier. Although the mean-variance criterion predicts that a lowering of the expected 3-year return from 19% to 15% allows for a lower standard deviation, the CVaR criterion predicts that this lowering of the return expectation leads in fact to an increase of the risk (as measured by CVaR). In terms of asset allocation, CVaR shows that lowering the proportion of emerging market bonds below 20% is not rewarded by a risk reduction.
Figure 7 Efficient frontiers obtained using mean-variance (green) and mean-CVaR (blue) optimization. Top graph shows both efficient frontiers as returns vs. standard deviation and bottom graph shows the same efficient frontiers as returns vs. CVaR. Optimization is performed on historical monthly data compounded on a rolling window of 12 months. $\alpha = 0.95$. 
Figure 8 Asset allocation for the efficient frontiers plotted in Figure 7. Portfolios are numbered for each frontier from left (least risky) to right (most risky). Note that portfolios with the same number obtained using different optimization methods may correspond to different Return and/or Risk.
Figure 9 Efficient frontiers obtained using mean-variance (green) and mean-CVaR (blue) optimization. Top graph shows both efficient frontiers as returns vs. standard deviation and bottom graph shows the same efficient frontiers as returns vs. CVaR. Optimization is performed on historical monthly data compounded on a rolling window of 36 months. $\alpha = 0.95$. 
Figure 10 Asset allocation for the efficient frontiers plotted in Figure 9. Portfolios are numbered for each frontier from left (least risky) to right (most risky). Note that portfolios with the same number obtained using different optimization methods may correspond to different Return and/or Risk.
Overall, the differences between Markowitz and CVaR efficient frontiers are found to be minimal when optimization is performed on historical distributions or rolling window compounded return distributions. Notable differences in asset allocation are results of rather small changes in the risk or return on the efficient frontier. A possible cause of this behavior is that assets with larger departure from normality, like high-yield or emerging market bonds, constitute a small fraction of the portfolio. Therefore, a big change in the allocation to these assets is required in order to change the risk-return profile of the overall portfolio.
3.2 Monte Carlo Optimization Using the Skewed Student-T Distribution

When using historical data to perform portfolio optimization one encounters two major drawbacks. The first is that the past behavior does not necessarily reflect the future behavior of the asset prices. The second is that the amount of data on which the optimization is performed is limited by the size of the available time-series. This is a big problem when using low frequency data, i.e., one year or longer periods. These limitations can be avoided if one uses a synthetic method of generating the return distributions, namely a Monte-Carlo method. I have used such a technique in order to generate multivariate return distributions. Since CVaR optimization reaches its full potential in the case of non-normally distributed returns, I have used such a method based on the Skewed Student-T probability distribution.

The Skewed Student-T Monte-Carlo simulation method, still at an experimental stage, was developed “in house” by Merrill Lynch Bank (Suisse) S.A. For a description of the method see the Internship Report of Andrei (2005). I will not present here the details of this method; however, it suffices to say that it allows one to generate multivariate distributions that have the imposed values of the first four moments (mean, variance, skewness, and kurtosis) as well as the desired covariance matrix for the series generated. The parameters that I used are a combination of historical values (variance/covariance matrix, skewness and kurtosis) and forecasted parameters (means). The investment universe in this case is formed of: stocks, U.S government bonds, alternative investments (hedge funds), and commodities. The historical parameters used were extracted from monthly data on four indices that represent the four asset classes:

- MSCI: MSCI World Index TR (% Total Return)
- USGv: Citigroup U.S. 1+ Yr Gvt TR (% Total Return)
- HF: HFRI FoF Index (% Total Return)
- COM: Goldman Sachs Commodity Index (% Total Return)

The time series for the indices start on January 1994 and end in March 2005. Using the Monte-Carlo technique I have generated new time series representing monthly returns with the parameters listed in Table 4. For yearly returns I have used independently generated monthly data and I have compounded 12 of them to obtain the yearly return series that have the parameters shown in Table 5.
Table 4 Statistical properties of the Monte-Carlo generated monthly series used for portfolio optimization. The multivariate Monte-Carlo simulation is based on the Skewed Student-T distribution. Sample size: 120.

<table>
<thead>
<tr>
<th>Statistical Properties</th>
<th>MSCI</th>
<th>USGv</th>
<th>HF</th>
<th>COM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0076</td>
<td>0.0037</td>
<td>0.0053</td>
<td>0.0037</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0403</td>
<td>0.0231</td>
<td>0.0252</td>
<td>0.0641</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.2027</td>
<td>0.0194</td>
<td>-0.8812</td>
<td>0.7492</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.9920</td>
<td>2.7483</td>
<td>8.7094</td>
<td>4.1759</td>
</tr>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>4.50%</td>
<td>6.50%</td>
<td>4.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>15.22%</td>
<td>8.35%</td>
<td>9.26%</td>
<td>23.39%</td>
</tr>
</tbody>
</table>

Table 5 Statistical properties of the Monte-Carlo generated yearly series used for portfolio optimization. This series is obtained compounding 12 independent monthly returns Sample size: 120.

<table>
<thead>
<tr>
<th>Statistical Properties</th>
<th>MSCI</th>
<th>USGv</th>
<th>HF</th>
<th>COM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0950</td>
<td>0.0450</td>
<td>0.0650</td>
<td>0.0450</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.1619</td>
<td>0.0826</td>
<td>0.1043</td>
<td>0.2799</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.2878</td>
<td>0.1770</td>
<td>0.3500</td>
<td>1.2584</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.7737</td>
<td>3.5799</td>
<td>3.4046</td>
<td>5.3331</td>
</tr>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>4.50%</td>
<td>6.50%</td>
<td>4.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>16.19%</td>
<td>8.26%</td>
<td>10.43%</td>
<td>27.99%</td>
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</tbody>
</table>

<table>
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<tr>
<th>Correlation Coefficients</th>
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<th>USGv</th>
<th>HF</th>
<th>COM</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI</td>
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<td>0.2060</td>
<td>0.5426</td>
<td>-0.0072</td>
</tr>
<tr>
<td>USGv</td>
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<td>0.1479</td>
</tr>
<tr>
<td>HF</td>
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<td>1.0000</td>
<td>0.2540</td>
</tr>
<tr>
<td>COM</td>
<td>-0.0072</td>
<td>0.1479</td>
<td>0.2540</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
The multivariate non-normal distributions generated using the Monte-Carlo technique constituted the input of the CVaR optimization routine. The goal was once more to spot the differences between the results of mean-variance and mean-CVaR optimization regarding the efficient frontiers and the efficient asset allocations. These results are presented in Figures 11 and 12 for the monthly data and in Figures 13 and 14 for the yearly data. In both cases I have used a confidence level of 95% for evaluation of the CVaR.

Figures 11 to 14 show results that are similar to those obtained using historical distributions. In this case, the commodities have a rather small contribution to the formation of the overall portfolios, consequence of its rather poor overall performance. Interesting that CVaR yields a slightly higher proportion of investment in commodities since it detects a higher diversification benefit than mean-variance. Hedge funds receive an important fraction of the investment according to both optimization criteria due to their good risk-return characteristic. One has to notice that assets with return distributions that are close to normal still have the largest share in the asset allocation.

The differences in the efficient frontiers seen in Figures 11 and 13 are relatively small, not surpassing 0.5% in expected return for any fixed level of risk. The most notable feature appears in the case of the yearly data and is the extension of the mean-variance frontier towards the minimum variance portfolio, beyond the minimum CVaR portfolio. A decrease of the expected return from 6.2% to 5.2%, instead of being rewarded by a reduction of the risk as predicted by mean-variance analysis, leads to an increase in risk as measured by CVaR. This statement is reflected in asset allocation terms by saying that the reduction of the amount invested in equity, at the expense of the bond investment, does not reduce CVaR below the level of 0.75%.
Figure 11 Efficient frontiers obtained using mean-variance (green) and mean-CVaR (blue) optimization. Top graph shows both efficient frontiers as returns vs. standard deviation and bottom graph shows the same efficient frontiers as returns vs. CVaR. Optimization is performed on monthly data simulated using a Monte Carlo technique. Statistics of the asset distributions are shown in Table 4. ($\alpha = 0.95$).
Figure 12 Asset allocation for the efficient frontiers plotted in Figure 11. Portfolios are numbered for each frontier from left (least risky) to right (most risky). Note that portfolios with the same number obtained using different optimization methods may correspond to different Return and/or Risk.
Figure 13 Efficient frontiers obtained using mean-variance (green) and mean-CVaR (blue) optimization. Top graph shows both efficient frontiers as returns vs. standard deviation and bottom graph shows the same efficient frontiers as returns vs. CVaR. Optimization is performed on Monte Carlo simulated data for a buy-and-hold strategy with a 12 month investment horizon. Statistics of the asset distributions are shown in Table 5. (α = 0.95)
Figure 14 Asset allocation for the efficient frontiers plotted in Figure 13. Portfolios are numbered for each frontier from left (least risky) to right (most risky). Note that portfolios with the same number obtained using different optimization methods may correspond to different Return and/or Risk.
The conclusion of the single period optimization using a mean-CVaR frame is that differences from the usual mean-variance allocation are noticeable. CVaR is able to capture features of the distributions that go beyond the variance measurement of risk and changes the asset allocation accordingly. However, since the assets with more evidently non-normal distributions have a relatively small share of the investment, according to either optimization method, the resulting efficient frontiers are not very far apart. The situation is different when it comes to the actual asset allocation prescriptions given by each method. Here one can observe significant differences in the weights of the different assets. These differences stem from an over-determination of the efficient frontier. For Monte-Carlo simulated data, as well as for historical distributions, enough assets were included such that very close to efficient portfolios can be obtained using a wide range of asset allocation choices.
4. Two Period Constrained Portfolio Optimization

4.1 Long Term Portfolio Optimization with a Short Term Constraint

Consider a long term investor whose views on the short term behavior of the market differ from the long term ones. The dilemma he is facing is whether the portfolio construction should be made based on the long term predictions on the capital market behavior or on the short term ones. Ideally the portfolio should be constructed with the account of both. In what follows I explore the use of the CVaR optimization method for imposing dual constraints on the expected return distribution.

The problem formulated here is that of a portfolio manager who optimizes the portfolio of his client over a long term horizon aiming at high expected returns with the minimum risk taken. However, short term predictions of the “in house” analysts are significantly different from the long term ones. The manager would like to avoid the situation of a significant drop in the value of the portfolio in the short run, which might turn away the client. The idea is to use the short term market predictions in order to manage the short-term risk of the portfolio. The analysis is similar in nature to the risk management techniques employed by most financial institutions, however, it differs in the time horizon over which risk is controlled and in the measure of risk used (I use CVaR instead of VAR). This approach is similar to the Asset Liability Management approach using CVaR used by Bogentof et al. (2001). They consider a series of periods over which certain inflows and outflows occur in the portfolio. CVaR is used to minimize the probability of failing to make the required payments. Here, I only use two time horizons: a close term, over which there is a tactical view on the market and a long term, for which a strategic view of the market is available. In both approaches the optimization is performed based on the whole distribution and not conditional on the evolution of the market on each individual scenario. Note that I consider the case of only one scenario bundle in the sense of Bogentof et al. (2001).

The idea employed here is very simple. I construct two portfolio return distributions based on short and long term predictions of the asset price evolution. The portfolio optimization using the long term distribution can be performed as before. In addition, a
constraint on the level of CVaR calculated using the short term distribution can be imposed on the portfolio that is optimized. The formulation of the problem in mathematical terms is:

\[
\begin{align*}
\text{max}_{\omega^1, \omega^2, \pi} & \quad \sum_{i=1}^{N} w_i \cdot E[R_i(t_1)] \\
\text{s.t} & \quad a^1 - \frac{1}{1-\alpha} \sum_{j=1}^{J} z_j^1 \leq \text{CVaR}_1, \\
& \quad z_j^1 \geq -w_i \cdot R_i(t_1) - a^1, \\
& \quad z_j^1 \geq 0, \\
& \quad a^2 - \frac{1}{1-\alpha} \sum_{j=1}^{J} z_j^2 \leq \text{CVaR}_2, \\
& \quad z_j^2 \geq -w_i \cdot R_i(t_2) - a^2, \\
& \quad z_j^2 \geq 0, \\
& \quad w_i \geq 0, \\
& \quad \sum_{i=1}^{N} w_i = 1.
\end{align*}
\]

Here \(R_i(t_{1,2})\) are the returns of the asset \(i\) in scenario \(j\) at time \(t_1\) or \(t_2\). \(\text{CVaR}_2\) is the minimum CVaR constraint allowed at time \(t_2\). This value is varied such that the whole \(t_2\) efficient frontier is constructed. On the other hand, \(\text{CVaR}_1\) is the maximum CVaR allowed at time \(t_1\). This parameter keeps a fixed value for the construction of an entire (constrained) efficient frontier based on the return–risk values at time \(t_2\). \(\text{CVaR}_1\) represents the maximum risk that the investor is ready to accept on the short term while still investing for the best results over long-term.

The MATLAB code which implements this optimization is listed in the Appendix. At the core of the solution is again the MATLAB-provided linear programming routine. Note the size of the problem has increased to \(2J + N + 1\), roughly double when compared to the single horizon case. I consider here the case where both distributions have the same size: \(J\), but this is not a requirement in general. The size of each sampling can be chosen such that it best describes the assumed distribution for the short, respectively long, time intervals.
4.2 Capital Market Assumptions for Different Investment Horizons

The first thing one has to note when using the framework defined above is that nontrivial results are conditioned by having different expectations on the market behavior for different time frames. If the market expectations on the short term and on the long term are identical then any CVaR condition for the short term will have a one-to-one correspondence to a CVaR value imposed on the long term distribution. The only way in which the short term condition affects the long-term efficient frontier is by limiting its extension beyond that value of \( CVaR_2 \) that corresponds to the imposed value of \( CVaR_1 \).

As an example of two-period constrained CVaR optimization I have considered the case of a log-normal distribution for each of the assets considered. The fact that the distributions for the two investment horizons are characterized by means and variance-covariance matrices that do not scale linearly with time, is enough to guarantee non-trivial results. Note that for a one period investment, normality assumptions lead to trivial results of CVaR, meaning that the results coincide with those of traditional mean-variance optimization. In the case of the two-period optimization, I label as trivial those results for which the efficient frontier does not differ in shape from that obtained through a one period optimization. This happens if the market assumptions for short and long term horizons coincide. The difference in market assumptions guarantees a non-trivial result for the example considered below, however, it does not imply that these results would be different from those obtained using a similar two period conditioned mean-variance optimization. Nevertheless, to my knowledge, such type of two-period optimization has not been performed using either CVaR or mean-variance criteria.

I consider here the same investment universe as in the previous section: equity, US government and high yield bonds, hedge funds, and commodities. The long term market assumptions are again based on the predictions of the Merrill Lynch analysts (Merrill Lynch, 2005). Multivariate normal distributions are generated using the means, volatilities and correlation matrices shown in Table 6. Short term predictions represent a more delicate issue. A strong view on these values implies the assumption that there are significant inefficiencies in market pricing of certain assets. The assumptions that I made here are just a test-tube case for the purpose of exemplification of the optimization method. They do not reflect the exact
opinions of any analyst at this moment. The way I arrived to the numbers listed in Table 7, which parameterize the short term distributions used, is by making a weighted average between the long term predictions (66%) and the historic values obtained using monthly data of the last 3 years (33%). These assumptions represent a replacement for the investor’s short term views on the market.

Table 6 Assumptions about returns, volatilities and correlation matrices for a 10 year investment horizon

<table>
<thead>
<tr>
<th>Market Assumptions</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>HF</th>
<th>COM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized Return</td>
<td>9.50%</td>
<td>5.00%</td>
<td>6.50%</td>
<td>7.00%</td>
<td>5.00%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>18.00%</td>
<td>3.00%</td>
<td>7.50%</td>
<td>8.00%</td>
<td>13.00%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>MSCI</th>
<th>USGv</th>
<th>HYB</th>
<th>HF</th>
<th>COM</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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</tr>
<tr>
<td>HYB</td>
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<td>0.4134</td>
<td>-0.0112</td>
</tr>
<tr>
<td>HF</td>
<td>0.3677</td>
<td>0.1845</td>
<td>0.4134</td>
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<td>0.1372</td>
</tr>
<tr>
<td>COM</td>
<td>0.0384</td>
<td>-0.1527</td>
<td>-0.0112</td>
<td>0.1372</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 7 Assumptions about returns, volatilities and correlation matrices for a 1 year investment horizon

<table>
<thead>
<tr>
<th>Market Assumptions</th>
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<th>USGv</th>
<th>HYB</th>
<th>HF</th>
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<tr>
<td>Annualized Return</td>
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<td>5.50%</td>
<td>9.00%</td>
<td>7.00%</td>
<td>9.50%</td>
</tr>
<tr>
<td>Annualized Volatility</td>
<td>17.00%</td>
<td>3.00%</td>
<td>8.00%</td>
<td>7.00%</td>
<td>14.00%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>MSCI</th>
<th>USGv</th>
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<th>HF</th>
<th>COM</th>
</tr>
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<tbody>
<tr>
<td>MSCI</td>
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<td>-0.2806</td>
<td>0.3990</td>
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<td>0.0769</td>
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<td>USGv</td>
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<td>0.0994</td>
<td>0.0868</td>
<td>0.2855</td>
</tr>
<tr>
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<td>1.0000</td>
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<td>0.0044</td>
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<tr>
<td>HF</td>
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<td>0.0868</td>
<td>0.4674</td>
<td>1.0000</td>
<td>0.4284</td>
</tr>
<tr>
<td>COM</td>
<td>0.0769</td>
<td>0.2855</td>
<td>0.0044</td>
<td>0.4284</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
4.3 Constrained Portfolio Optimization Results

Using the above two sets of parameters, I have generated two multivariate normal distributions: one for a 1 year time interval and another for a 10 year time interval. These distributions represent the continuously compounded returns; the total 1 or 10-year returns follow shifted log-normal distributions since prices are log-normal. Figures 15 and 16 represent the efficient frontiers obtained separately, using either of the two distributions. These results are identical, up to statistical errors, to mean-variance optimizations based on the same assumptions. The asset allocations for the efficient portfolios reflect exactly the different market assumptions that are used. The example that is presented involves a very bullish short-term view on commodities and a bearish short-term view on hedge funds. Over long term I have considered trends that are reversed. These views are reflected in the absence of the hedge funds in the short term portfolio while a notable amount is allocated to commodities. On the other hand, the long-term frontier contains a large allocation to hedge funds and nothing visible to commodities. Next I describe how these two different allocations can be combined into a single one.

Remember that $CVaR_1$ stands for the CVaR calculated using the distribution after 1 year and $CVaR_2$ for the one calculated using the 10 year distribution. Observe that $CVaR_1$ values spanned by the 1 year efficient frontier range from -2.5% (average net gain even in the worst cases) and up to 14% for the riskiest portfolios. These values reflect the whole spectrum of risk levels incurred by an investment made for any period longer than one year. As a result, any of these values can be imposed as a limit of the risk taken after only one year. The least restrictive condition is $CVaR_1 = 14\%$. This condition practically does not restrict at all the long term optimization; therefore, in this case, the “constrained” optimization is identical to the unconstrained one. At the opposite extreme, imposing $CVaR_1 = -2.5\%$ drastically reduces the freedom of allocation and yields a constrained frontier that is very different from the unconstrained one. Intermediate values of the $CVaR_1$ lay in-between these two extremes.
Figure 15 Efficient frontier and efficient portfolio composition for the 10 year (long term) investment horizon. Returns are normally distributed with parameters listed in Table 6.
Figure 16 Efficient frontier and efficient portfolio composition for the one year (short term) investment horizon. Returns are normally distributed with parameters listed in Table 7.
**Figure 17** Constrained efficient frontiers for 10 year allocations. A maximum CVaR constraint is imposed after one year. The values indicated on the figure are those of the 1 year CVaR. Constrained efficient portfolio compositions are shown in Figure 18.
**Figure 18** Constrained efficient portfolio composition corresponding to the efficient frontiers shown in Figure 17. The color-asset correspondence is the same as indicated in Figures 15 and 16.
I present in Figure 17 the unconstrained and five constrained efficient frontiers corresponding to five values of $CVaR_i$. Notably the curves of the constrained efficient frontiers match exactly the unconstrained frontier for a certain portion of the graph after which they turn sharply to the right. The corresponding asset allocations are shown in Figure 18. $CVaR_i = 12$ is very close to the unconstrained allocation along most of the efficient frontier. On the contrary, the low $CVaR_i$ optimal allocations contain visibly less hedge funds and more commodities, as a direct consequence of the short term assumptions made. The results show the gradual change of the portfolio allocation as more stringent restrictions are imposed on $CVaR_i$. It is important to note that in the most restrictive case, the constrained efficient allocation does not become the same as the short-term efficient allocation. The reason is that the optimization is performed through the maximization of long-term returns mean for a fixed long-term CVaR. The short term distribution is used only to impose an additional constraint; however, the short-term mean is not directly involved in the optimization process. This is fundamentally different from the short-term mean-CVaR allocation. A further extension of the analysis presented here, could combine the long- and short-term risk-and-return such that a smooth transition between the two efficient portfolios is obtained.
5. Conclusions

The present work is based on a recently introduced method of portfolio optimization which measures the investment risk using conditional value at risk. This method has already been used in some academic studies and has started to make its entrance in the financial industry. This method appears to be a very promising alternative portfolio construction technique to both the traditional mean-variance optimization and to the value at risk minimization. The main reasons why this method is preferable are:

i) a more accurate account of the risky side of the assumed return distribution,

ii) flexibility: it can be used in conjunction with almost any method of generating expected distribution of asset returns.

iii) wide applicability: it can be used easily on very particular financial instruments such as structured products and alternative investments where the traditional mean-variance optimization method fails.

As part of this work I have implemented, using the MATLAB software package, the CVaR optimization method using the linear programming function of the MATLAB Optimization Toolbox. The linear programming routine limits the number of scenarios that I could use, when compared to other examples found in literature. These examples cite the use of other commercially available linear optimization libraries that considerably improve the optimization time.

The main purpose of the CVaR optimization was the attempt to reveal a few particular aspects related to the role of the investment horizon in the process of strategic asset allocation. In this context I have performed portfolio optimization using asset return series generated by two different methods: historic series but with means shifted according to our expectations of the future market returns, and Monte Carlo generated series using a Skewed Student-T distribution in a multivariate context. In both cases I have used traditional assets with distributions very close to normal and also distributions with visible departure from normality, like hedge funds or emerging market bonds. However, I have not used any non-linear payoff instruments like options or more general structured products. Using both return series, I have compared the CVaR and the mean-variance generated efficient frontiers for
different investment horizons when using a buy-and-hold strategy. The differences that I have found are relatively small, much less than 10% of the existing levels of risk or return in all cases. The reason why changes in the position of the efficient frontier are so small could be the fact that, usually the strongly non-normal assets are also quite risky, so they represent a small part in the optimal portfolio. Since the non-normally distributed assets are a small part of the portfolio, the resulting frontier is quite close to the mean-variance frontier. However, the allocations on these frontiers can differ significantly, mostly because I have considered enough assets in order to have many different allocations which are very close to optimum according to either optimization criterion. In general, differences are higher for monthly data, where the departure from normality of some series is more evident. Buy-and-hold investments of one year or more yield differences that are even smaller.

The central point of this work is the generalization of CVaR portfolio optimization for the case of a dual objective investor. I consider the problem of simultaneously accounting for the long and short term market views of an investor. The main optimization objective is maximization of the long term expected return while taking limited levels of risk both on the long and short term. The best application of this optimization method is the private banking industry where certain individuals are interested in the value maximization over the long term, but display also a considerable level of risk aversion over short term. The result of this optimization is most interesting when the short and long term expectations on market behavior are very different. The method presented here can give the composition of the optimal portfolio conditioned by the short term risk not being larger than predefined levels. It is worth mentioning that the results of this optimization could not be of better quality than the assumptions made about the short term prices evolution. Making accurate predictions about market behavior is very difficult. Also, such predictions are generally in contradiction with the efficient market hypothesis. However, long term expectations about the economy may be considerably different than the short term expectations which are included in the current price. In this situation the method is useful to all investors who want to minimize their short to medium term risk, regardless of their beliefs about the efficiency of specific markets.
References


Appendix

MATALB functions that perform CVaR optimization on a single distribution of expected returns. This code uses the `linprog()` linear programming routine, part of the Optimization Toolbox of MATLAB 7.0.0 (R14), 2004.

```matlab
function pweights = cvaroptir(IReturn,ScenRets,beta)
    %This function performs a CVaR minimization while a minimum
    %Return constraint is imposed
    
    % Inputs:
    % ScenRets: Matrix of size J x N containing the N asset returns
    % for the J scenarios considered (this matrix samples the
    % actual return distribution of the assets considered)
    % IReturn: The minimum Expected Return for the portfolio
    % beta: The confidence level used to calculate CVaR
    
    % Outputs:
    % pweights: vector of size N containing the weights of the assets
    % in the optimal portfolio

    % Find Size of the problem: No. of assets and No of scenarios
    [Jscn Nasst] = size(ScenRets);
    
    % Function to be minimized
    ff = [1 zeros(1,Nasst) ones(1,Jscn)/Jscn/(1-beta)];
    ff = ff';

    % Inequalities on the dummy variables z
    % L.H.S.
    AA = [-ones(Jscn,1) -ScenRets -eye(Jscn)];
    % R.h.s.
    bb = zeros(1+Jscn,1);

    % Inequality on expected return
    % L.H.S.
    AA = [AA; [0 -mean(ScenRets(:,:)) zeros(1,Jscn)]]; % R.h.s.
    bb(1+Jscn,1) = -IReturn;

    % Lower Bounds
    lb = zeros(1+Nasst+Jscn,1);
    lb(1,1) = -100;

    % Equality (portfolio consistancy)
    Aeq = [0 ones(1,Nasst) zeros(1,Jscn)];
    beq = [1];
    tic
    % Call LinProg Subroutine
    [xx, fval, exitflag, output] = linprog(ff, AA, bb, Aeq, beq, lb, [], [], optimset('LargeScale','on','Simplex','off','Display','off'));
```

56
function pweights = cvaroptcvcv(ICVaR,ScenRets,beta)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%This function performs a Return maximization while a maximum
%CVaR constraint is imposed
%
% Inputs:
% ScenRets:  Matrix of size J x N  containing the N asset returns
% for the J scenarios considered (this matrix samples the
% actual return distribution of the assets considered)
% ICVaR:      The maximum CVaR for the portfolio
% beta:       The confidence level used to calculate CVaR
%
% Outputs:
% pweights:   vector of size N containing the weights of the assets
% in the optimal portfolio
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Find Size of the problem: No. of assets and No of scenarios
[Jscn  Nasst]=size(ScenRets);

%Function to be minimized
ff=[0 -mean(ScenRets(:,:)) zeros(1,Jscn)/Jscn/(1-beta)];
ff=ff';

%Inequalities on the dummy variables z
%L.H.S.
AA=[-ones(Jscn,1) -ScenRets -eye(Jscn)];
%R.h.s.
bb=zeros(1+Jscn,1);

%Inequality on CVaR
%L.H.S.
AA=[AA;[1 zeros(1,Nasst) ones(1,Jscn)/Jscn/(1-beta)]];
%R.h.s.
bb(1+Jscn,1)=ICVaR;

%Lower Bounds
lb=zeros(1+Nasst+Jscn,1);
lb(1,1)=-100;

%Equality (portfolio consististency)
Aeq=[0 ones(1,Nasst) zeros(1,Jscn)];
beq=[1];

%Call LinProg Subroutine
[xx,fval,exitflag,
 output]=linprog(ff,AA,bb,Aeq,beq,lb,[],[];,optimset('LargeScale','on','Simpl
ex','off',
 'Display','off'));
if exitflag < 1
    disp(output);
    disp(['No Assets = ' num2str(Nasst) ' No Scenarios = ' num2str(Jscn)]);
    return
end
pweights=xx(2:Nasst+1)'
End

MATLAB function that performs a simultaneous CVaR-optimization on two distributions of expected returns. This code uses the *linprog()* linear programming routine, part of the Optimization Toolbox of MATLAB 7.0.0 (R14), 2004.

function pweights = cvaropticv2(ICVaR1,ScenRets1,ICVaR2,ScenRets2,beta)

% This function performs a Return maximization while 2 maximum CVaR constraints are imposed using two different return distributions
% Inputs:
%       ScenRets1: Matrix of size J x N containing the N asset returns for the J scenarios considered at time t1
%       ScenRets2: Matrix of size J x N containing the N asset returns for the J scenarios considered at time t2
% NOTE: Return over THIS distribution is maximized
% NOTE: ScenRets1 and ScenRets2 must have the same dimensions, this is not checked
% ICVaR1: The maximum CVaR for the portfolio over distribution at t1
% ICVaR2: The maximum CVaR for the portfolio over distribution at t2
% beta: The confidence level used to calculate CVaR1 and CVaR2
% Outputs:
%       pweights: vector of size N containing the weights of the assets in the optimal portfolio
%
% Find Size of the problem: No. of assets and No of scenarios
[Jscn Nasst]=size(ScenRets1);

% Construct LP Problem
% Function to be minimized
ff=[-mean(ScenRets2(:,:)) 0 zeros(1,Jscn)/Jscn/(1-beta) 0 zeros(1,Jscn)/Jscn/(1-beta)];
ff=ff';

% Inequalities on the dummy variables z
L.H.S.
AA1=[-ScenRets1 -ones(Jscn,1) -eye(Jscn) zeros(Jscn,1) zeros(Jscn) ];
AA1=[AA1; [zeros(1,Nasst) 1 ones(1,Jscn)/Jscn/(1-beta) 0 zeros(1,Jscn) ]];
% R.h.s.
bb1=zeros(Jscn,1);
bb1=[bb1; ICVaR1];
% L.H.S.
AA2=[-ScenRets2 zeros(Jscn,1) zeros(Jscn) -ones(Jscn,1) -eye(Jscn)];
AA2=[AA2; [zeros(1,Nasst) 0 zeros(1,Jscn) 1 ones(1,Jscn)/Jscn/(1-beta) ]];
% R.h.s.
bb2=zeros(Jscn,1);
bb2=[bb2; ICVaR2];

AA=[AA1; AA2];
bb=[bb1;bb2];

% Lower Bounds
lb=[zeros(Nasst,1);-100;zeros(Jscn,1);-100;zeros(Jscn,1)];

% Equality (portfolio consistency)
Aeq=[ones(1,Nasst) 0 zeros(1,Jscn) 0 zeros(1,Jscn)];
beq=[1];
% tic
% Call LinProg Subroutine
[xx,fval,exitflag, output]=linprog(ff,AA,bb,Aeq,beq,lb,[],[],... optimset('LargeScale','on','Simplex','off','Display','off'));
if exitflag < 1
    disp(output);
    disp(['No Assets = ' num2str(Nasst) ' No Scenarios = ' num2str(Jscn)]);
    return
end
pweights=xx(1:Nasst)';
end